Deduction of the Power Series Representing a Function from Special Values of the Latter.

By G. W. Hill.

I have already treated this matter in another place,* but the exposition there is by illustration only and quite incomplete. The subject needs a more general presentation, which will be the endeavor here.

The treatment of the question is much facilitated or, in many cases, even rendered possible, by the application of two principles. The first is the isolation of groups in the assemblage of linear equations through the attribution of zero values to some of the parameters involved. The second is the disintegration of the equations by comparison when corresponding positive and negative values are given to one or more of the parameters.

Here it is expedient to adopt a peculiar notation. Let F denote the function to be treated and x the general parameter. The formulæ to be written in what follows will be limited to the case where there are four parameters; the modifications to be made when there are more or less will be obvious. We use i for the general integral exponent always not negative, and A for the general coefficient. There is here no necessity for the employment of accents or subscripts to distinguish quantities of the same kind. The parameters will be known as the first, second, third and fourth. In designating any one of these all must be written; thus, the third parameter is $x^0x^0xx^0$. Accordingly, we write the equation

$$F=\Sigma Ax^ix^ix^ix^i,$$

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where the i's are not necessarily the same. Let subscripts attached to F denote the special values of the function correspondent to special values of the parameters; and, as we have to distinguish between significant and zero values for the latter, let us suppose that i always denotes a positive integer; consequently, the value $i = 0$ is excluded from the summations Σ. The function F must undergo a sort of differencing in reference to zero values for the parameter; a differencing which is more general than the ordinary, since it often involves more than one variable. This mode of operating is, for the adopted case, depicted in the following system of equations:

$$\Sigma \,.\, Ax^0x^0x^0x^0 = F_{0000} = \overset{0}{F}_{0000},$$

$$\left\{\begin{aligned}
&\Sigma \,.\, Ax^ix^0x^0x^0 = F_{x000} - F_{0000} = \overset{1}{F}_{x000},\\
&\Sigma \,.\, Ax^0x^ix^0x^0 = F_{0x00} - F_{0000} = \overset{1}{F}_{0x00},\\
&\Sigma \,.\, Ax^0x^0x^ix^0 = F_{00x0} - F_{0000} = \overset{1}{F}_{00x0},\\
&\Sigma \,.\, Ax^0x^0x^0x^i = F_{000x} - F_{0000} = \overset{1}{F}_{000x},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
&\Sigma \,.\, Ax^ix^ix^0x^0 = F_{xx00} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00} - F_{0000} = \overset{2}{F}_{xx00},\\
&\Sigma \,.\, Ax^ix^0x^ix^0 = F_{x0x0} - \overset{1}{F}_{x000} - \overset{1}{F}_{00x0} - F_{0000} = \overset{2}{F}_{x0x0},\\
&\Sigma \,.\, Ax^ix^0x^0x^i = F_{x00x} - \overset{1}{F}_{x000} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{x00x},\\
&\Sigma \,.\, Ax^0x^ix^ix^0 = F_{0xx0} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - F_{0000} = \overset{2}{F}_{0xx0},\\
&\Sigma \,.\, Ax^0x^ix^0x^i = F_{0x0x} - \overset{1}{F}_{0x00} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{0x0x},\\
&\Sigma \,.\, Ax^0x^0x^ix^i = F_{00xx} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{00xx},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
&\Sigma \,.\, Ax^ix^ix^ix^0 = F_{xxx0} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x0x0} - \overset{2}{F}_{0xx0} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00}\\
&\qquad\qquad\qquad\qquad - \overset{1}{F}_{00x0} - F_{0000} = \overset{3}{F}_{xxx0},\\
&\Sigma \,.\, Ax^ix^ix^0x^i = F_{xx0x} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x00x} - \overset{2}{F}_{0x0x} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00}\\
&\qquad\qquad\qquad\qquad - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{xx0x},\\
&\Sigma \,.\, Ax^ix^0x^ix^i = F_{x0xx} - \overset{2}{F}_{x0x0} - \overset{2}{F}_{x00x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{x000} - \overset{1}{F}_{00x0}\\
&\qquad\qquad\qquad\qquad - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{x0xx},\\
&\Sigma \,.\, Ax^0x^ix^ix^i = F_{0xxx} - \overset{2}{F}_{0xx0} - \overset{2}{F}_{0x0x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0}\\
&\qquad\qquad\qquad\qquad - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{0xxx},
\end{aligned}\right.$$

$$\Sigma\,.\,Ax^ix^ix^ix^i = F_{xxxx} - \overset{3}{F}_{xxx0} - \overset{3}{F}_{xx0x} - \overset{3}{F}_{x0xx} - \overset{3}{F}_{0xxx} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x0x0}$$
$$- \overset{2}{F}_{x00x} - \overset{2}{F}_{0xx0} - \overset{2}{F}_{0x0x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{x000}$$
$$- \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} \qquad = \overset{4}{F}_{xxxx}\,.$$

The number of these equations is $16 = 2^4$, and, generally, if there are k parameters, the number is 2^k. It will be readily perceived that $\overset{0}{F}$ is the term of the series independent of the parameters; that the $\overset{1}{F}$ are functions of the single significant parameter appearing in their subscripts, without a term independent of that parameter; that the $\overset{2}{F}$ are functions of the two significant parameters appearing in their subscripts, without any terms independent of one or both parameters; that the $\overset{3}{F}$ are functions of the three significant parameters appearing in their subscripts, without any terms independent of one, two or all of these parameters; and, finally, that $\overset{4}{F}$ is a function of all four parameters, but without any terms independent of one, two, three or all of the parameters. Thus each F of a definite superscript involves no terms included in the F of smaller superscripts. By this device we have broken the system of linear equations for the determination of the coefficients into 16 groups, each of which can be treated independently of the others.

It is not necessary that the computations should be made by the equations just written. The last involves no less than 16 terms, and labor will be saved by eliminating some of the F. The 5 equations at the beginning remaining unmodified, it will be perceived the following system is equivalent to the former :

$$\begin{cases} \overset{2}{F}_{xx00} = F_{xx00} - \overset{1}{F}_{x000} - F_{0x00}, \\ \overset{2}{F}_{x0x0} = F_{x0x0} - \overset{1}{F}_{00x0} - F_{x000}, \\ \overset{2}{F}_{x00x} = F_{x00x} - \overset{1}{F}_{000x} - F_{x000}, \\ \overset{2}{F}_{0xx0} = F_{0xx0} - \overset{1}{F}_{00x0} - F_{0x00}, \\ \overset{2}{F}_{0x0x} = F_{0x0x} - \overset{1}{F}_{000x} - F_{0x00}, \\ \overset{2}{F}_{00xx} = F_{00xx} - \overset{1}{F}_{00x0} - F_{000x}, \end{cases}$$

$$\left\{\begin{aligned}
\overset{3}{F}_{xxx0} &= F_{xxx0} - \overset{2}{F}_{xx00} - F_{x0x0} - F_{0xx0} + F_{00x0}, \\
\overset{3}{F}_{xx0x} &= F_{xx0x} - \overset{2}{F}_{xx00} - F_{0x0x} - F_{x00x} + F_{000x}, \\
\overset{3}{F}_{x0xx} &= F_{x0xx} - \overset{2}{F}_{00xx} - F_{x00x} - F_{x0x0} + F_{x000}, \\
\overset{3}{F}_{0xxx} &= F_{0xxx} - \overset{2}{F}_{00xx} - F_{0x0x} - F_{0xx0} + F_{0x00},
\end{aligned}\right.$$

$$\overset{4}{F}_{xxxx} = F_{xxxx} - \overset{3}{F}_{xxx0} - \overset{3}{F}_{xx0x} - \overset{2}{F}_{xx00} - F_{x0xx} - F_{0xxx} + F_{00xx}.$$

These formulæ are not the unique ones of their type, but the $\overset{2}{F}$ admit two different forms, the $\overset{3}{F}$ three and $\overset{4}{F}$ six. All are obtained by making certain transpositions between the x and 0 of the subscripts. They need not be given here, as their employment has no advantage over those just written.

Each of the $\overset{i}{F}$ is evidently divisible by the product of the significant parameters in its subscript. The functions thus obtained may be considered as one step nearer the result of elimination. We may use G to denote them. Thus:

$$G_{0000} = \frac{1}{x^0x^0x^0x^0} F_{0000}, \qquad G_{x000} = \frac{1}{xx^0x^0x^0} \overset{1}{F}_{x000}, \qquad G_{0x00} = \frac{1}{x^0xx^0x^0} \overset{1}{F}_{0x00}, \text{ etc.,}$$

$$G_{xx00} = \frac{1}{xxx^0x^0} \overset{2}{F}_{xx00}, \text{ etc.,} \qquad G_{xxx0} = \frac{1}{xxxx^0} \overset{3}{F}_{xxx0}, \text{ etc.,} \qquad G_{xxxx} = \frac{1}{xxxx} \overset{4}{F}_{xxxx}.$$

We come now to the application of the second principle. In the first place consider F when involving only a single significant parameter as F_{x000}, and let F_{+000} and F_{-000} denote the values of F_{x000} for corresponding positive and negative values of x; then it is plain we shall have

$$\begin{aligned}
\Sigma A\,(x^2)^i\,x^0x^0x^0 &= \tfrac{1}{2}\,[F_{+000} + F_{-000}], \\
xx^0x^0x^0\,\Sigma A\,(x^2)^i\,x^0x^0x^0 &= \tfrac{1}{2}\,[F_{+000} - F_{-000}],
\end{aligned}$$

where the A of the first equation are distinct from the A of the second, and where it is now necessary to allow i to assume the value 0.

Next, supposing F involves two significant parameters as F_{xx00}, then we shall have the four equations

$$\begin{aligned}
\Sigma A\,(x^2)^i\,xx^0x^0 &= \tfrac{1}{2}\,[F_{+x00} + F_{-x00}], \\
xx^0x^0x^0\Sigma A\,(x^2)^i\,xx^0x^0 &= \tfrac{1}{2}\,[F_{+x00} - F_{-x00}], \\
\Sigma Ax(x^2)^i\,x^0x^0 &= \tfrac{1}{2}\,[F_{x+00} + F_{x-00}], \\
x^0xx^0x^0\Sigma Ax\,(x^2)^i\,x^0x^0 &= \tfrac{1}{2}\,[F_{x+00} = F_{x-00}].
\end{aligned}$$

By taking half the sum and half the difference of certain of these, we obtain the four equations

$$\begin{aligned}
\Sigma A x^{2i} x^{2i} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} + F_{+-00} + F_{-+00} + F_{--00}\right],\\
\Sigma A x^{2i+1} x^{2i} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} + F_{+-00} - F_{-+00} - F_{--00}\right],\\
\Sigma A x^{2i} x^{2i+1} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} - F_{+-00} + F_{-+00} - F_{--00}\right],\\
\Sigma A x^{2i+1} x^{2i+1} x^0 x^0 &= \tfrac{1}{4}\left[F_{++00} - F_{+-00} - F_{-+00} + F_{--00}\right],
\end{aligned}$$

where the coefficients A are distinct for each. The second, third and fourth are divisible severally by $xx^0x^0x^0$, $x^0xx^0x^0$ and xxx^0x^0. By making these divisions we shall be a step nearer the result of elimination. The rule of the signs connecting the form F in the group of four equations may seem a little obscure, but a consideration of the successive operations of taking half the sum and difference shows that the sign of each F is given by raising the signs in the subscripts to the same powers as the corresponding parameters have in the left members of the equations. As here, the even integer $2i$ may be dropped out of the exponents, we perceive that the signs in question are given by the expressions

$$\begin{array}{llll}
(+)^0(+)^0, & (+)^0(-)^0, & (-)^0(+)^0, & (-)^0(-)^0,\\
(+)^1(+)^0, & (+)^1(-)^0, & (-)^1(+)^0, & (-)^1(-)^0,\\
(+)^0(+)^1, & (+)^0(-)^1, & (-)^0(+)^1, & (-)^0(-)^1,\\
(+)^1(+)^1, & (+)^1(-)^1, & (-)^1(+)^1, & (-)^1(-)^1.
\end{array}$$

In case F involves three significant parameters, as F_{xxx0}, we have entirely analogous equations which, for brevity, we write as follows:

$$\begin{aligned}
\Sigma A x^{2i} x^{2i} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^0(\pm)^0 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i+1} x^{2i} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^0(\pm)^0 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i} x^{2i+1} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^1(\pm)^0 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i} x^{2i} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^0(\pm)^1 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i+1} x^{2i+1} x^{2i} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^1(\pm)^0 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i+1} x^{2i} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^0(\pm)^1 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i} x^{2i+1} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^0(\pm)^1(\pm)^1 F_{\pm\pm\pm 0},\\
\Sigma A x^{2i+1} x^{2i+1} x^{2i+1} x^0 &= \tfrac{1}{2^3} S(\pm)^1(\pm)^1(\pm)^1 F_{\pm\pm\pm 0}.
\end{aligned}$$

The connection of the ambiguous signs in these equations will be readily understood.

In case F has all its parameters significant, there are 16 equations analogous to the preceding; we need write but one as a type of all:

$$\Sigma A x^{2i} x^{2i} x^{2i} x^{2i} = \tfrac{1}{2^4} S (\pm)^0 (\pm)^0 (\pm)^0 (\pm)^0 F_{\pm\pm\pm\pm}.$$

Thus, by the application of the two principles of zero values and of pairs of values of opposite signs, we succeed in breaking the system of linear equations to be solved into several subordinate systems entirely independent of each other. When the number of parameters is 2, it is evident that the number of these subordinate systems is

$$1.1 + 2.2 + 4.1 = 9 = 3^2;$$

where there are 3 parameters this number is

$$1.1 + 2.3 + 4.3 + 8.1 = 27 = 3^3;$$

and when the number of parameters is 4 (the case we have been treating) the number is

$$1.1 + 2.4 + 4.6 + 8.4 + 16.1 = 81 = 3^4;$$

hence, in the general case, where there are k parameters, the number of independent subordinate systems is 3^k.

After having shown the applicability of zero and parity values of the parameters for breaking the system of linear equations into detached portions, it remains to show what principles should guide us in selecting the values of the parameters for which the special values of the function are to be computed. As in the former memoir we suppose that the values of each parameter are taken from an arithmetical progression of which one term is zero. Let d denote the common difference in this progression which, although it may be different for each parameter, we designate by the same letter, just as before we employed x and i. Then, in the first instance, the power series will be derived in the form

$$F = \Sigma A \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i.$$

As it is necessary to cut off the power series at some limit, it is desirable to choose the d in such a way that the neglected terms should vitiate as little as possible the derived values of the A. The smaller are the d the smaller is this vitiation; but practical considerations set a limit to this diminution. Suppose we are going to quantities of the 10^{th} order of smallness in the x, and decide to halve the d; then, as $2^{10} = 1024$, it will be necessary to add 3 more decimals in our computations; and if the d are diminished to a tenth, 10 decimals must be added, which procedure could not generally be entertained. Thus nice judgment is required in deciding on the magnitude of the d. As good a rule for the choice as can be given is to divide the range over which the parameter is supposed to play by the number of significant exponents it is to receive in the power series. Then the selection of the values of the parameters should be such that, in a graphical exhibition, they would be arranged as nearly as possible in a symmetrical manner about the origin; and, in a space of k dimensions if k is the number of parameters, they should be contained within the ellipsoid whose axes are the several ranges.

As the computation of special values of the function constitutes much the larger part of the labor incident to the method, it is desirable to insist on the limitation that no more special values are to be computed than terms are to be retained. But some restrictions must be put on the employment of the two principles given for the purpose of disintegrating the system of linear equations, and on the selection of values of the parameters for which the special values of the function are to be computed.

If F is computed for $x = id$, $x = id$, $x = id$, $x = id$, we shall call $iiii$ the argument of the value of F. Here i is integral, but may be zero or negative. Then, in each group of linear equations obtained by the application of the first principle, it is plain that the zeros of the arguments used must fall in the same place as the zero exponents of the parameters; thus, when we are treating the group whose type is $[i00i]$, the arguments of the special values used must be of the type $i00i$, where, however, i can be negative. The first principle can always be used, but it is desirable to limit the selection of arguments in the following manner:—Dividing the terms into Division I, where all the exponents are zeros, Division II, when all but one are zeros, and III where all but two are zeros, and so on; if we have used an argument such as $iii0$ in Division IV, it is necessary to use the arguments $ii00$, $i0i0$, $0ii0$ in the preceding or here Division III,

understanding that that the i in the second case are identical with those standing in the same place in the first. Hence the proper method of selecting the arguments to be used seems to be to commence at Division I, for which, in the case we exhibit, the argument is 0000, and get the arguments for Division II by substituting for one of the zeros an integer positive or negative. Then the arguments for Division III are got from these by substituting for one of the remaining zeros positive or negative integers, and so on to the end. These integers should constitute in each case an arithmetical progression having zero near the middle of it.

With regard to the application of the second principle, that of parity values it often cannot be employed without introducing non-independent equations. The remedy for this state of things is to cut down the operations to a half stage or even to a quarter stage, and, in some cases, not to employ it at all.

These matters cannot be well set forth without the help of an example. We adopt that of the preceding memoir. It is characterized by saying that it involves four parameters, two of which are regarded as of the first, and two of the second order of smallness; and all terms above the eighth order are to be neglected. This demands the presence of 175 terms in the power series. How they are disintegrated into 81 subgroups by the application of our two principles is shown in the following table. As each term is sufficiently characterized by the exponents of the four parameters, nothing else is set down, and the terms of each subgroup are connected by the sign $+$. In addition, the exponents of the first term are set down as they are, but the following terms of the line are divided by the first term, as the quotients are more useful than the terms themselves.

Division.	Group.	Sub-group.	Divisor.	Quotients.
I	1	1	0000	
II	2	2	2000	+2000+4000+6000
		3	1000	“ “ “
	3	4	0200	+0200+0400+0600
		5	0100	“ “ “
	4	6	0020	+0020
		7	0010	“
	5	8	0002	+0002
		9	0001	“
III	6	10	2200	+2000+0200+4000+2200+0400
		11	1200	“ “ “ “ “
		12	2100	“ “ “ “ “
		13	1100	+ “ “ “ “ +6000 +4200+ 2400 +0600
	7	14	2020	“
		15	1020	“
		16	2010	“ +0020 “
		17	1010	“ “ “
	8	18	2002	“
		19	1002	“
		20	2001	“ +0002 “
		21	1001	“ “ “
	9	22	0220	+0200
		23	0120	“
		24	0210	“ +0020+0400
		25	0110	“ “ “
	10	26	0202	“
		27	0102	“
		28	0201	“ +0002 “
		29	0101	“ “ “
	11	30	0022	
		31	0012	
		32	0021	
		33	0011	+0020 ‘

Division.	Group.	Sub-group.	Divisor.	Quotient.					
		34	2220						
		35	1220						
		36	2120						
	12	37	2210	+2000	+0200				
		38	1120	"	"				
		39	1210	"	"				
		40	2110	"	"				
		41	1110	"	"	+0020	+4000	+2200	+0400
		42	2202						
		43	1202						
		44	2102						
	13	45	2201	"	"				
		46	1102	"	"				
		47	1201	"	"				
IV		48	2101	"	"				
		49	1101	"	"	+0002	"	"	"
		52	2012						
		53	2021						
	14	54	1012						
		55	1021						
		56	2011	"					
		57	1011	"					
		60	0212						
		61	0221						
	15	62	0112						
		63	0121						
		64	0211	+0200					
		65	0111	"					
		76	2211						
		77	2111						
V	16	78	1211						
		79	1121						
		80	1112						
		81	1111	+2000	"				

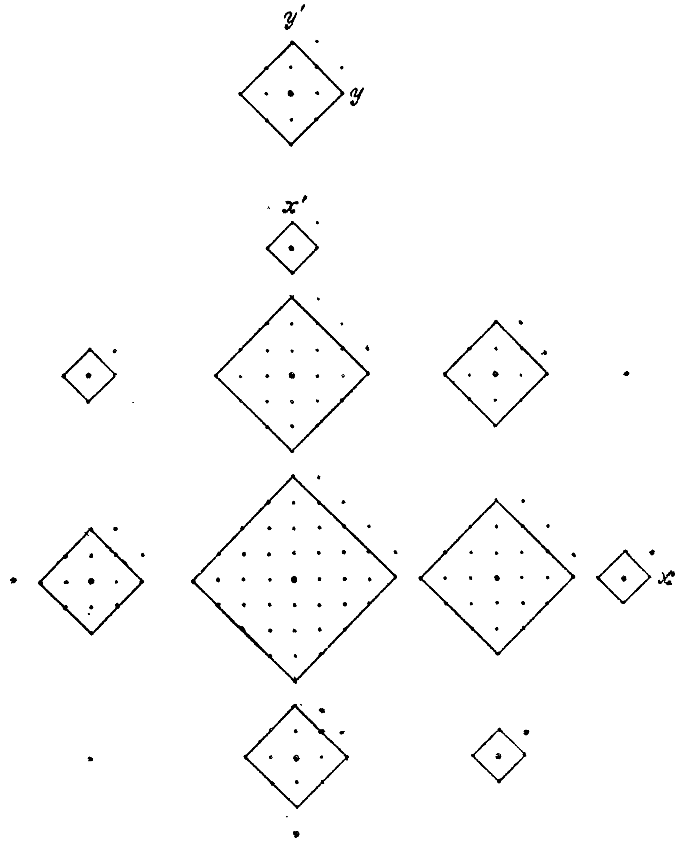
The 16 groups in the table are the result of the application of the first principle; the 81 sub-groups result from the further application of the second principle. It will be noticed that 14 out of the 81 sub-groups do not appear in the table; this is because their terms are all above the 8^{th} order. The success of the application of the two principles is well shown by the table. Out of 81 sub-groups there is only one (the 13^{th}) which consists of as many as 10 equations and 10 unknowns; two groups have 7, and three have 6; and 23 groups consist of a single equation giving the value of one coefficient each.

The following table shows a selection of arguments which may be employed in our illustrative example. The ambiguous signs must be taken in every possible combination; thus three in one argument denote eight different arguments.

Division.	Group.	Arguments.									
I	1	0 0 0 0									
II	2	±1 0 0 0	±2 0 0 0	±3 0 0 0	±4 0 0 0						
	3	0 ±1 0 0	0 ±2 0 0	0 ±3 0 0	0 ±4 0 0						
	4	0 0 ±1 0	0 0 ±2 0								
	5	0 0 0 ±1	0 0 0 ±2								
III	6	±1 ±1 0 0	±2 ±1 0 0	±1 ±2 0 0	±2 ±2 0 0	±3 ±1 0 0	±1 ±3 0 0	1 4 0 0	2 3 0 0	3 2 0 0	4 1 0
	7	±1 0 ±1 0	±2 0 ±1 0	±1 0 2 0	±3 0 1 0						
	8	±1 0 0 ±1	±2 0 0 ±1	±1 0 0 2	±3 0 0 1						
	9	0 ±1 ±1 0	0 ±2 ±1 0	0 ±1 2 0	0 ±3 1 0						
	10	0 ±1 0 ±1	0 ±2 0 ±1	0 ±1 0 2	0 ±3 0 1						
	11	0 0 ±1 ±1	0 0 2 1	0 0 1 2							
IV	12	±1 ±1 ±1 0	±2 ±1 1 0	±1 ±2 1 0	2 1 −1 0	1 2 −1 0	2 2 1 0	3 1 1 0	1 3 1 0	1 1 2 0	
	13	±1 ±1 0 ±1	±2 ±1 0 1	±1 ±2 0 1	2 1 0 −1	1 2 0 −1	2 2 0 1	3 1 0 1	1 3 0 1	1 1 0 2	
	14	±1 0 ±1 1	±1 0 1 −1	±2 0 1 1							
	15	0 ±1 ±1 1	0 ±1 1 −1	0 ±2 1 1							
V	16	±1 ±1 1 1	1 1 −1 1	1 1 1 −1	2 1 1 1	1 2 1 1					

It will be seen from this table that the second principle has not in every case been pushed to its limit. Thus in Div. IV, Group 12, if we employ the 8 arguments $\pm 1 \pm 1 \pm 1\ 0$ we get as many independent relations between the sought

coefficients; but, if we annex the 8 augments $\pm 2 \pm 1 \pm 1\ 0$, we do not get 8 additional relations but only 5. This is explained by the fact (consult the arrangement of terms in Group 12 in the first table) that the first 8 give the values of 3 coefficients, and the second 8 also give them.



We will catalogue all the deviations from a complete parity treatment in the foregoing table. In Group 6, the parity treatment, here involving two steps, has been applied only in six cases, while four arguments are without it; to have applied it to the latter would have introduced superfluous relations. In Groups 7-10 we have two instances of parity treatment to two steps, and two to one step. In Group 11 one instance of this treatment to two steps and two arguments without it. In Groups 12 and 13 one instance to three steps, two to two and six arguments without it. In Groups 14 and 15 one instance to two steps

and two to one. In fine, in Group 16 one instance to two steps and four arguments without it.*

But it is much easier to comprehend the principles which should be followed in the choice of the arguments through a graphical exhibition. The 175 arguments in our example, since they are to four elements, can be represented in a space of four dimensions. By drawing in this space $3.5 = 15$ planes properly chosen, the points representing the arguments will all lie in these planes. We adopt here for the coordinates the notation of the first memoir, viz., $xx'yy'$. In the adjacent diagram the upper oblique square with its two adjacent points constitutes a table of contents or index to the graphs of the 15 planes shown below; it bears on the coordinates y and y' or the third and fourth constituents of the argument. These graphs are placed relatively to each other as the points of the index which belong to them. By this device we are enabled to represent on a plane, sufficiently for our purposes, a space of four dimensions. Moreover, the graphs are placed so as not to interfere with each other, the coordinates x and x' being measured from the central point of the oblique squares. The introduction of the latter into the diagram has no other object than to enable the eye to grasp quickly the law of distribution of the points.

It will be perceived that 5 of the graphs reduce to a single point; they may be called oblique squares to side 0. Next 4 graphs consist of oblique squares to side 1 and they all have one point exterior to the square. Again there are 3 graphs to side 2 with 2 exterior points. Next 2 graphs to side 3 with 3 exterior points; and finally, a single graph to side 4 with 4 exterior points. With regard to these exterior points, it must be explained that the positions they have in the diagram are not unique. Let us suppose that the positions lying nearest the perimeter of an oblique square and exterior to it are called the adjacent points; they are in number four times the number expressing the side of the square, and they can be joined by straight lines so as to form rectangles. Then the exterior points must be distributed in such a way that each rectangle shall receive one and but one point at some one of its angles. It is not necessary that a similar arrangement should be adopted for all or for some of the graphs; it may be varied at will. In the diagram the exterior points are, in all cases, placed to the

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*The choice of arguments made here for the 175 special values of F is not quite the same as that in the first memoir. If, in the 8 arguments numbered 120-123, 142-145, 1 is substituted for 3 in the first constituent, we have the present selection.

upper and right side of the square. As to the arrangement of the squares in reference to the magnitude of their sides, it will be perceived that on the one hand the limit is a square of the side 0, and on the other a square of side 1; and, as we pass inwards towards the centre, at every step the side augments by 2; but when we arrive at the middle column, it is only a half-step on the right hand, while it is a whole step on the left. This is for an even number of parameters; for an odd number, the half steps do not exist.

The number of points in each graph is shown by the following scheme:

$$\left.\begin{array}{l} 1.1 \\ 1.1 + 2.3 \\ 1.1 + 2.3 + 3.5 \\ 1.1 + 2.3 + 3.5 + 4.7 \\ 1.1 + 2.3 + 3.5 + 4.7 + 5.9 \end{array}\right\} = 175.$$

The regularity apparent in the diagram is due to the tabulation of the points under the headings of two of the parameters. However, after the diagram is formed, it will not be difficult to distribute the arguments under the headings of the groups. It will be noticed that the exterior points are each a half-unit distant from the perimeters of the squares. As we have placed them they may be included in a rectangle having one more column in one direction than in the other.

When we have the arguments of the special values which determine the coefficients of a sub-group, it is easy to write, with the assistance of the first table, the determinant belonging to the solution. Thus, in Sub-group 13 of our example the determinant is

$$\begin{vmatrix}
1^0.1^0 & 1^2.1^0 & 1^0.1^2 & 1^4.1^0 & 1^2.1^2 & 1^0.1^4 & 1^6.1^0 & 1^4.1^2 & 1^2.1^4 & 1^0.1^6 \\
2^0.1^0 & 2^2.1^0 & 2^0.1^2 & 2^4.1^0 & 2^2.1^2 & 2^0.1^4 & 2^6.1^0 & 2^4.1^2 & 2^2.1^4 & 2^0.1^6 \\
1^0.2^0 & 1^2.2^0 & 1^0.2^2 & 1^4.2^0 & 1^2.2^2 & 1^0.2^4 & 1^6.2^0 & 1^4.2^2 & 1^2.2^4 & 1^0.2^6 \\
2^0.2^0 & 2^2.2^0 & 2^0.2^2 & 2^4.2^0 & 2^2.2^2 & 2^0.2^4 & 2^6.2^0 & 2^4.2^2 & 2^2.2^4 & 2^0.2^6 \\
3^0.1^0 & 3^2.1^0 & 3^0.1^2 & 3^4.1^0 & 3^2.1^2 & 3^0.1^4 & 3^6.1^0 & 3^4.1^2 & 3^2.1^4 & 3^0.1^6 \\
1^0.3^0 & 1^2.3^0 & 1^0.3^2 & 1^4.3^0 & 1^2.3^2 & 1^0.3^4 & 1^6.3^0 & 1^4.3^2 & 1^2.3^4 & 1^0.3^6 \\
4^0.1^0 & 4^2.1^0 & 4^0.1^2 & 4^4.1^0 & 4^2.1^2 & 4^0.1^4 & 4^6.1^0 & 4^4.1^2 & 4^2.1^4 & 4^0.1^6 \\
3^0.2^0 & 3^2.2^0 & 3^0.2^2 & 3^4.2^0 & 3^2.2^2 & 3^0.2^4 & 3^6.2^0 & 3^4.2^2 & 3^2.2^4 & 3^0.2^6 \\
2^0.3^0 & 2^2.3^0 & 2^0.3^2 & 2^4.3^0 & 2^2.3^2 & 2^0.3^4 & 2^6.3^0 & 2^4.3^2 & 2^2.3^4 & 2^0.3^6 \\
1^0.4^0 & 1^2.4^0 & 1^0.4^2 & 1^4.4^0 & 1^2.4^2 & 1^0.4^4 & 1^6.4^0 & 1^4.4^2 & 1^2.4^4 & 1^0.4^6
\end{vmatrix}$$

There is no need of proving that these determinants are non-vanishing, as they are all met with in the problem of drawing a parabolic curve through a definite number of distinct points in a space of two or more dimensions.